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On Laplace transforms of certain special functions

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1 Introduction

In this note we consider a Laplace transform of the following multiple integral

$$g(d;n;y)=c\pi^{d/2}\frac{1}{y^{\{n(d+1)-d\}/2+1}}\exp\{-\frac{a_{n}^{2}}{y}\}$$
$$=\int\cdots\int_{0\leq v_{1}+\cdots+v_{n-1}\leq 1;0\leq v_{j}}\exp\{\Sigma_{j=1}^{n-1}(a_{n}^{2}-a_{j}^{2})v_{j}/y\}$$
$$\left[v_{1}\cdots v_{n-1}(1-\Sigma_{j=1}^{n-1}v_{j})\right]^{(d+1)/2-1}dv_{1}\cdots dv_{n-1} \tag{1}$$

where c is a constant. This multiple integral is coming from the normed product of multidimensional Cauchy densities, namely

$$f(d;a_{1},\cdots a_{n};x)=\frac{c_{1}}{\pi_{j=1}^{n}(a_{j}^{2}+|x|^{2})^{(d+1)/2}},$$

where c_1 is the normalised constant and $0<a_{j};\ a_{j}\neq a_{k}$ if $j\neq k$, and $x=(x_{1},\cdots,x_{d})\in\mathbf{R}^{d}$. We note that the density function $g(d;1;u)$ can be expressed by a confluent hypergeometric function,

$$g(d;1;u) = \frac{c\pi^{d/2}\Gamma((d+1)/2)^{2}}{\Gamma(d+1)} \cdot\frac{1}{u^{d/2+2}}\exp\{-\frac{b^{2}}{u}\}\ {}_{1}F_{1}(\frac{d+1}{2};d+1;\frac{(b^{2}-a^{2})}{u}) \tag{2}$$

for all dimension d. For the dimension $d=3$ we obtain

$$g(3;1;u) = \frac{c\pi^{3/2}}{(b^{2}-a^{2})^{2}}\Big[\frac{1}{u^{3/2}}(\exp\{-\frac{a^{2}}{u}\}+\exp\{-\frac{b^{2}}{u}\}) -\frac{2}{(b^{2}-a^{2})u^{1/2}}(\exp\{-\frac{a^{2}}{u}\}-\exp\{-\frac{b^{2}}{u}\})\Big] \tag{3}$$

and for the dimension $d = 5$ we obtain

$$\begin{aligned} g(5;1;u) &= \frac{c\pi^{5/2}}{(b^2-a^2)^3}\Big[\frac{2!}{u^{3/2}}(\exp\{-\frac{a^2}{u}\} - \exp\{-\frac{b^2}{u}\}) \\ &\quad -\frac{2\cdot 3!}{(b^2-a^2)u^{1/2}}(\exp\{-\frac{a^2}{u}\} + \exp\{-\frac{b^2}{u}\}) \\ &\quad +\frac{4!u^{1/2}}{(b^2-a^2)^2}(\exp\{-\frac{a^2}{u}\} - \exp\{-\frac{b^2}{u}\})\Big]. \end{aligned} \tag{4}$$

2 The Laplace transforms of the case $n = 1$

Let us denote the Laplace transform of the functions $g(d;1;u)$ by $\eta(d;1;s)$ and let us take a branch of the value $\eta(d;+0) = 1$. We obtain

$$\begin{aligned} \eta(3;1;s) &= \frac{c\pi^2}{(b^2-a^2)^2}\Big\{(\frac{1}{a}\exp\{-2a\sqrt{s}\} + \frac{1}{b}\exp\{-2b\sqrt{s}\}) \\ &\quad -\frac{2}{(b^2-a^2)\sqrt{s}}(\exp\{-2a\sqrt{s}\} - \exp\{-2b\sqrt{s}\})\Big\}, \end{aligned} \tag{5}$$

for the dimension $d = 3$ and for the dimension $d = 5$ we obtain

$$\begin{aligned} \eta(5;1;s) &= \frac{c\pi^3}{(b^2-a^2)^3}[2!(\frac{1}{a}\exp\{-2a\sqrt{s}\} - \frac{1}{b}\exp\{-2b\sqrt{s}\}) \\ &\quad -\frac{2\cdot 3!}{(b^2-a^2)\sqrt{s}}(\exp\{-2a\sqrt{s}\} + \exp\{-2b\sqrt{s}\}) \\ &\quad +\frac{4!}{(b^2-a^2)^2 s}\{a\ \exp\{-2a\sqrt{s}\}(1+\frac{1}{2a\sqrt{s}}) \\ &\quad -b\ \exp\{-2b\sqrt{s}\}(1+\frac{1}{2b\sqrt{s}})\}\Big]. \end{aligned} \tag{6}$$

Making use of the formula

$$K_\nu(z) = \frac{1}{2}(\frac{1}{2}z)^\nu \int_0^\infty \exp\{-t - \frac{z^2}{4t}\}\ \frac{dt}{t^{\nu+1}}, \tag{7}$$

we obtain Laplace transforms for the odd dimensional case,

$$\begin{aligned} \eta(d;1;s) = \frac{c\pi^{d/2}}{\{(b^2-a^2)^{l+1}}\Sigma_{j=0}^{l}\frac{(-1)^{2l-j}(l+j)!}{(b^2-a^2)^j}\binom{l}{j}2^{j+1/2}\,. \\ \Big\{a^{2j-1}\frac{K_{(j-1)+1/2}(2a\sqrt{s})}{(2a\sqrt{s})^{j-1+1/2}} + (-1)^{l+1+j}b^{2j-1}\frac{K_{(j-1)+1/2}(2b\sqrt{s})}{(2b\sqrt{s})^{j-1+1/2}}\Big\}, \end{aligned} \tag{8}$$

where

$$K_{n+1/2}(z) = (\frac{\pi}{2z})^{1/2}\exp\{-z\}\Sigma_{r=0}^{n}\frac{(n+r)!}{r!(n-r)!(2z)^r} \tag{9}$$

(cf. G. Watson [7]).

3 Nonzero region of the Laplace transform

In general, we obtain

$$\eta(d;1;s) = c\pi^{l+1}\int_0^1 \frac{v^l(1-v)^l}{(b^2-(b^2-a^2)v)^{(d+2)/2}}$$
$$\exp\{-2(b^2-(b^2-a^2)v)^{1/2}\sqrt{s}\}$$
$$\Sigma_{r=0}^{l+1}\frac{(l+1+r)!}{r!(l+1-r)!2^r}\{(b^2-(b^2-a^2)v)^{1/2}\sqrt{s}\}^{l+1-r}dv \tag{10}$$

for the odd dimension $d = 2l+1$ and we see that $\eta(d;1;s) \neq 0$ in a neighborhood of the origin except at the origin and $\eta(d;1;s)$ is bounded in a neighborhood of the origin. From the above Laplace transforms (8) we can see that the Laplace transforms $\eta(d;1;s)$ do not vanish in a neighborhood of the infinity point.

4 The Laplace transform for the general case

Let us obtain the Laplace transform of $g(d;n;y)$ for the odd dimension. The following equality holds:

$$\exp\{-a_n^2 t\}\cdot t^{nl+n-1}$$
$$\cdot\int\cdots\int_{0\le v_1+\cdots+v_{n-1}\le 1;0\le v_j}\exp\Big\{\Sigma_{j=1}^{n-1}\Big(a_n^2-a_j^2\Big)tv_j\Big\}$$
$$\cdot\Big[v_1\cdots v_{n-1}\Big(1-\Sigma_{j=1}^{n-1}v_j\Big)\Big]^l dv_1\cdots dv_{n-1}$$
$$=(-1)^{(l+1)(n-1)}\Sigma_{r=1}^{n}\exp\{-a_r^2 t\}$$
$$\cdot\Big[\Sigma_{j_{n-1}=0}^{l}\frac{l!(l+j_{n-1})!}{j_{n-1}!(l-j_{n-1}+1)!}\frac{1}{B(r;n,n-1)^{l+j_{n-1}+1}}$$
$$\cdot\Sigma_{j_{n-2}=0}^{l-j_{n-1}}\frac{(l-j_{n-1})!(l+j_{n-2})!}{j_{n-2}!(l-j_{n-1}-j_{n-2})!}\frac{1}{B(r;n,n-2)^{l+j_{n-2}+1}}$$
$$\cdot\Sigma_{j_{n-3}=0}^{l-j_{n-1}-j_{n-2}}\frac{(l-j_{n-1}-j_{n-2})!(l+j_{n-3})!}{j_{n-3}!(l-j_{n-1}-j_{n-2}-j_{n-3})!}$$
$$\cdot\frac{1}{B(r;n,n-3)^{l+j_{n-3}+1}}\cdots$$
$$\cdot\Sigma_{j_1=0}^{l-j_{n-1}-j_{n-2}\cdots-j_2}\frac{(l-j_{n-1}-j_{n-2}-\cdots-j_2)!(l+j_1)!}{j_1!(l-j_{n-1}-j_{n-2}-\cdots-j_2-j_1)!}$$
$$\cdot\frac{1}{B(r;n,1)^{l+j_1+1}}t^{l-j_1-\cdots-j_{n-2}-j_{n-1}}\Big] \tag{11}$$

where $B(r;n,1) = a_r^2 - a_{r+1}^2, \cdots,\ B(r;n,n-r) = a_r^2 - a_n^2,\ B(r;n,n-r+1) = a_r^2 - a_1^2, \cdots,\ B(r;n,n-1) = a_r^2 - a_{r-1}^2$.

Theorem 1 *The Laplace transform is given in the following form.*

$$\eta(d;n;s) = \int_0^\infty \exp\{-sy\}\ g(d;n;y)dy$$
$$= c_1\pi^{l+1/2}(-1)^{(l+1)(n-1)}$$
$$\Sigma_{r=1}^{n}\Big[\Sigma_{j_{n-1}=0}^{l}\frac{l!(l+j_{n-1})!}{j_{n-1}!(l-j_{n-1}+1)!}\frac{1}{B(r;n,n-1)^{l+j_{n-1}+1}}$$
$$\Sigma_{j_{n-2}=0}^{l-j_{n-1}}\frac{(l-j_{n-1})!(l+j_{n-2})!}{j_{n-2}!(l-j_{n-1}-j_{n-2})!}\frac{1}{B(r;n,n-2)^{l+j_{n-2}+1}}$$
$$\Sigma_{j_{n-3}=0}^{l-j_{n-1}-j_{n-2}}\frac{(l-j_{n-1}-j_{n-2})!(l+j_{n-3})!}{j_{n-3}!(l-j_{n-1}-j_{n-2}-j_{n-3})!}$$
$$\frac{1}{B(r;n,n-3)^{l+j_{n-3}+1}}\cdots$$
$$\Sigma_{j_1=0}^{l-j_{n-1}-j_{n-2}\cdots-j_2}\frac{(l-j_{n-1}-j_{n-2}-\cdots-j_2)!(l+j_1)!}{j_1!(l-j_{n-1}-j_{n-2}-\cdots-j_2-j_1)!}$$
$$\frac{1}{B(r;n,1)^{l+j_1+1}}2\Big(\frac{2}{2a_r\sqrt{s}}\Big)^{\Sigma_{p=1}^{n-1}j_p-1/2}K_{\Sigma_{p=1}^{n-1}j_p-1/2}\big(2a_r\sqrt{s}\big)\Big] \tag{12}$$

Here $K_{\Sigma_{p=1}^{n-1}j_p-1/2}\big(2a_r\sqrt{s}\big)$ *is given by (9).*

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